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A categorical setting for the 4-Colour Theorem

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Abstract

The 4-Colour Theorem has been proved in the late seventies (Appel and Haken, 1977; Appel et al., 1977), after more than a century of fruitless efforts. But the proof has provided very little new information about the map colouring itself. While trying to understand this phenomenon, we analyze colouring in terms of universal properties and adjoint functors.

It is well known that the 4-colouring of maps is equivalent to the 3-colouring of the edges of some graphs. We show that every slice of the category of 3-coloured graphs is a topos. The forgetful functor to the category of 3-coloured graphs is cotripleable; every loop-free graph is covered by a 3-coloured one in a universal way. In this context, the 4-Colour Theorem becomes a statement about the existence of a coalgebra structure on graphs.

In a sense, this approach seems complementary to the known combinatorial colouring procedures.

1. Introduction: The meaning of the Four Colours

A (planar) *map* is usually defined to be a particular embedding of a bridgeless graph in a plane. And a *graph* consists of two sets: the set V of *vertices* and E of *edges*. They are connected by an *incidence* function, which assigns to each edge an unordered pair $\{a, b\}$ of vertices. In principle, the vertices are represented as points, the edges as line segments. A *loop* is an edge incident with a single vertex (i.e., with the pair $\{a, a\} = \{a\}$). *A bridge* is an edge the removal of which disconnects the graph (in the topological sense). A graph is *planar* if it can be embedded in a plane, i.e., drawn in such a way that two edges intersect only at a common vertex. The circuits of adjacent edges may divide the plane into regions, which are often called *faces* of the graph. Two different edges are said to be *adjacent* if they have a common vertex; an edge is adjacent to itself if it is a loop. Two different faces are adjacent if they have a common edge; a face is adjacent to itself if at least one of its boundaries is a bridge.

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An infinite region (every finite planar graph must leave one) also counts as a face. In fact, drawing graphs in a plane and on a sphere boils down to the same thing – the only difference being that, in the latter case, all faces are finite. Graph theory makes no distinction between planar and spheric graphs – nor between planar and spheric maps. It does distinguish, however, the graphs which cannot be embedded in a plane or on a sphere.

To *n-colour* a map means to partition its faces in n disjoint classes in such a way that adjacent faces come in different classes. Clearly, the fewer classes you allow, the more difficult this task becomes. It is easy to construct maps which cannot be 3-coloured. The 4-colour theorem asserts, of course, that all planar maps[1] can be 4-coloured.

This was conjectured by a student Francis Guthrie in 1852, and publicized as a question in the *Proceedings of the London Mathematical Society* by Cayley in 1878. A year later, A.B. Kempe came with a very nice proof (which brought him fellowship in the *Royal Society*). The matter seemed settled until 1890, when Heawood detected an error in Kempe's proof, and went on to study Guthrie's conjecture for the rest of his life. (Without this error, would anybody still remember Guthrie, Kempe, Heawood and the Four Colours?) The tantalizing simplicity of the map colouring became one of the main driving forces behind the subsequent development of graph theory. And yet, the original conjecture resisted all efforts for another hundred years. By the nineteen seventies, the problem of graph colouring on all kinds of surfaces was completely solved, only the planar case withstood the developed techniques. Some of them provided quite deep informations, but not enough. Unexpectedly, the problem gave way to an approach that did not seek an understanding of complexities, but managed to avoid it!

The idea was to list an *unavoidable set* of configurations, some of which would have to appear in any minimal counterexample to the 4-Colour Conjecture; and to show that all these configurations can be *reduced*, so as to yield a yet smaller counterexample, in contradiction with the minimality assumption. The most persistent proponent of this idea was H. Heesch, who spent many years (now we are in the sixties) reducing configurations and finding larger and larger unavoidable sets. He hoped that the process would stop somewhere, producing a finite list of reducible configurations, which no minimal counterexample could avoid. But when would it stop? Both the reductions and the list grew out of sight.

This is where computers enter the scene. The calculations here seem to be lengthy even for them. When Heesch and Dürre wrote a program to reduce an unpleasant configuration, it ran for 26 hours. Nevertheless, Shimamoto announced in 1971 that he had proved the 4-Colour Conjecture, using Heesch's ideas and a computer. It took a paper of two prominent graph theorists, Whitney and Tutte [18], to explain what was wrong with this proof and with this whole approach. But in 1977, Appel and

[1] Usually, just the finite ones are considered: if there is an infinite counterexample, then there must be a finite one too.

Haken (with Koch) published an account [2, 3] on how they really proved the 4-Colour Conjecture – using Heesch's ideas and a computer. The computer ran for 1200 hours, the unavoidable set contained almost 2000 reducible configurations, with no perceivable regularity among them – but it "caught the mouse".

Since then, some efforts has been spent on chasing and correcting errors in this proof [4]. The unavoidable set has dropped to 1476 configurations (and the authors hope that it could be reduced to about 1200). The full list has been published in 1989 [5]. Amusing accounts of the whole history of 4-colouring have been written [1, 16]. But somehow, it seems that the mathematical community is showing much less enthusiasm for the solution than it used to show for the problem.[2]

In fact, the existing solution confronts us with a new problem, perhaps more fundamental than the original one. From the beginning, mathematics was built as a dialogue of formal proofs and intuitive explanations. They always went together, protecting each other from fallacy. And here comes *the* problem of Four Colours: there are ways to understand it, but none of them seems to be leading to a proof; and there is that formal proof which yields no understanding. Clearly, there is an open question here – not so much about maps, as about mathematics. If mathematics is not just a sequence of problems and solutions, but rather a processes of understanding, then Four Colours still deserve attention.

Actually, the embarrassment by the fact that computerized bookkeeping has solved something that a century of concentrated conceptual effort could not – seems to have concealed a crucial question. Everybody knows that there are physical processes too complex for structural insight. Unable to grasp them, we take rescue, say, in statistics. Do such processes come about even in the world of mathematics? How can one recognize them? The 4-Colour Theorem is a very interesting case. Is it uncovering an inherent limitation of conceptualisation in mathematics; or a threshold of complexity which just waits to be crossed? – Either way, a further analysis of the relation of the 4-colouring and the existing mathematical methods is needed.

This paper reports on an effort in this direction. What happens when the 4-Colour Problem is exposed to the categorical paradigm of conceptual mathematics, and vice versa? Our main goal was to describe the 4-colouring of maps as a universal construction – i.e., as an adjoint functor. In this way, one would hope to isolate the conceptual contents of a colouring procedure, and to recognize the source and sink of complexity. Using a well-known reduction of the map colouring to the edge colouring of cubic graphs, we provide an algebraic presentation of coloured graphs – which then yields the desired adjunction. A surprising feature of the result is that a colouring of a graph is not induced from smaller graphs – as in all combinatorial recipes – but projected from larger, though easily coloured graphs.

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[2] The volume devoted to Graph Theory in the prominent *Encyclopedia of Mathematics and its Applications* [17] (published in 1984) mentions the 4-Colour Theorem in a single sentence, despite the fact that its author W.T. Tutte used to be one of the main researchers of the problem in its time.

'A categorical characterisation of the Four Colour Theorem' in a rather different, purely combinatorial setting has been provided by Fawcett [10]. He shows that this theorem is equivalent to the surjectiveness of epimorphisms in a certain category of graphs. I shall try to relate Fawcett's approach with the present one in [15].

2. From 4-colouring to 3-colouring

A graph is *cubic* if it has exactly 3 edges incident with each vertex; a cubic map is induced by a cubic graph.

One of the very first facts about map colouring is that

all maps can be 4*-coloured if and only if the cubic ones can.*

Indeed, an arbitrary map can be made cubic by adding a new land at each vertex where more than three lands meet. A colouring of this new map yields a colouring of the old one: just remove the added lands. No new common boundaries between lands will be created and the colouring will remain correct. (Some lands which did not meet at all in the cubic map will meet at a vertex now; but this does not make them adjacent.)

Our treatment of map colouring is based on a further reduction, due to P.G. Tait. Already in the first years of the history of the 4-Colour Conjecture, he pointed out that

a cubic map can be 4*-coloured if and only if the edges of its underlying graph can be* 3*-coloured.*

This is not obvious, but it is fairly easy to prove. For instance, denote the four colours by the elements of the Kleinian group $K = \{0, i, j, k\}$. (Each element added to itself yields 0. The sum of two different nonzero elements is equal to the third one.) Now, if two adjacent lands in a cubic map are coloured by x and y, their boundary must be coloured by $x + y$. This gives an edge colouring in i, j, k. By reversing this process, one obtains the map colouring from an edge colouring: just choose a land to start from, colour it arbitrarily, and tour the world. Whenever you leave a land coloured in x across a border of the colour y, assign to the new land colour $x + y$. Thus, the 4-Colour Theorem is equivalent to the statement that

every bridgeless, cubic, planar graph can be edge 3*-coloured.*

Tait actually proposed a stronger conjecture – dropping the planarity assumption from this statement. But Tait's conjecture turned out to be false. *Petersen's graph,* shown in Fig. 1 cannot be edge 3-coloured, although it is cubic and contains no bridges.

Of course, this graph cannot be drawn on a sphere (but only on Klein's bottle). We shall later see how it can be "approximated" by a 3-colourable graph.

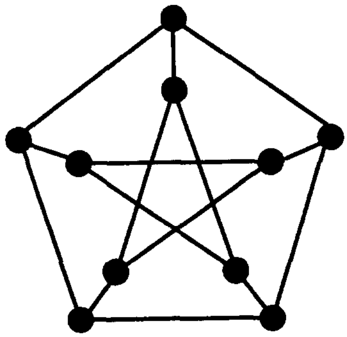

Fig. 1.

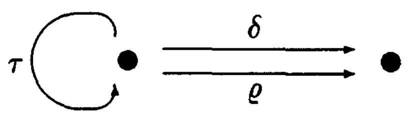

Fig. 2.

3. On toposes of graphs

A *topos* is a categorical model of set theory. (There are plenty of introductions to Topos Theory now: e.g., [8, 11, 12].) In this paper, we shall consider some *toposes of presheaves*. A *presheaf* is a (contravariant) functor to the category of sets. Persheaf toposes inherit their structure from this category. Various categories of graphs are presheaf toposes.

An *oriented* graph is again a pair of sets: V carries the vertices, E the edges. But the incidence relation now assigns an *ordered* pair of vertices to each edge, and this order is the orientation. In other words, an oriented graph is a pair of functions from E to V: they assign to each edge a head and a tail. Every such structure is the image in the category of sets of a unique functor from the category with two objects and two parallel arrows between them. Therefore, oriented graphs constitute a presheaf topos.

Ordinary (unoriented) graphs too. The simplest way to see this is to transform them into oriented graphs: replace each unoriented edge between the vertices x and y by a pair of oriented edges, one going from x to y, the other from y to x. In this way, ordinary graphs exactly correspond to those oriented graphs in which each edge has an "inverse". Such a graph can be represented as functor to sets from the category shown in Fig. 2, where $\delta \circ \tau = \varrho$ and $\tau^2 = id$. Natural transformations between these functors correspond to graph morphisms, which map vertices to vertices and edges to edges, while preserving the incidence.

This *topos of graphs* will be denoted by $\mathscr{G}$. Of course, nothing will be lost if we now go back from functors to the ordinary picture of graphs: the notion of topos is invariant under change of presentation.

4. Topos of colour algebras

A *colour algebra* is defined by unary operations $\mathbf{r}$, $\mathbf{b}$ and $\mathbf{g}$, which are involutive i.e. satisfy,

$$\begin{aligned} \mathbf{r}^2(x) &= x, \\ \mathbf{b}^2(x) &= x, \\ \mathbf{g}^2(x) &= x. \end{aligned} \tag{1}$$

Let $\mathscr{A}$ be the variety of colour algebras. Clearly, $\mathscr{A}$ can also be viewed as the topos of presheaves (i.e., actions) of the group Υ, with $\mathbf{r}$, $\mathbf{b}$ and $\mathbf{g}$ as the generators, satisfying the equations $\mathbf{r}^2 = 1$, $\mathbf{b}^2 = 1$ and $\mathbf{g}^2 = 1$. Acting on itself, this group appears in the topos $\mathscr{A}$ as the only representable presheaf $\hat{\Upsilon}$ – and a projective generator. On the other hand, Υ can be regarded as the free colour algebra over one generator.

We shall freely move between the two views of $\mathscr{A}$ – as a variety and as a topos – and usually neglect the difference between Υ and $\hat{\Upsilon}$.

The 3-coloured cubic graphs appear as those colour algebras where the inequalities

$$\begin{aligned} \mathbf{r}(x) &\neq x \\ \mathbf{b}(x) &\neq x \\ \mathbf{g}(x) &\neq x \end{aligned} \tag{2}$$

hold for all x. In fact, *all* colour algebras can be represented as cubic graphs: think of their elements as vertices, and put an edge between x and y if and only if $\mathbf{r}(x) = y$, or $\mathbf{b}(x) = y$, or $\mathbf{g}(x) = y$. In other words, there is a "forgetful" functor

$$U : \mathscr{A} \longrightarrow \mathscr{G}. \tag{3}$$

It maps a colour algebra A to a graph $G = UA$, where the underlying set of A becomes the set of vertices V_G, while the edges are the elements of

$$E_G := \{\{\mathbf{c}, x, y\} \mid \mathbf{c}(x) = y,\ \mathbf{c} \in \{\mathbf{r}, \mathbf{b}, \mathbf{g}\}\}. \tag{4}$$

The incidence function assigns the pair $\{x, y\}$ to the edge $\{\mathbf{c}, x, y\}$.

Now, if A satisfies (2), we can get a correct 3-colouring of G by taking $\mathbf{c} \in \{\mathbf{r}\text{ed}, \mathbf{b}\text{lue}, \mathbf{g}\text{reen}\}$ to be the colour of the edge $\{\mathbf{c}, x, y\}$. On the other hand, if any of the inequalities (2) is not satisfied, the graph G must have a loop and cannot be edge-coloured.

Therefore, the full subcategrory $\mathscr{C} \hookrightarrow \mathscr{A}$, spanned by colour algebras for which inequalities (2) hold, can be regarded as the *category of* 3-*coloured cubic graphs*. All the possible ways to 3-colour the edges of a cubic graph – not necessarily planar – are contained in it as objects. Its arrows are the colour-preserving graph morphisms. (The reader can easily check this.)

In the sequel, we shall often abbreviate the "edge 3-coloured cubic graphs" to "3-colourings".

5. Partial topos of 3-colourings

The next step is based on the trivial observation that a colour algebra A must satisfy (2) whenever there is a morphism $f: A \to C$ in $\mathscr{A}$, and C satisfies (2). In other words, for every 3-colouring $C \in \mathscr{C}$, the slice category $\mathscr{C}/C$ will be all of the category $\mathscr{A}/C$. Thus, every slice of $\mathscr{C}$ is a topos. This is what we mean when we say that $\mathscr{C}$ is a *partial topos.*[3] It inherits from $\mathscr{A}$ all the local structure.

As for the global structure, $\mathscr{C}$ is closed in $\mathscr{A}$ under products, coproducts and exponentiation. Indeed,

$$A \in |\mathscr{A}|,\ C \in |\mathscr{C}| \Rightarrow A \times C,\ C^A \in |\mathscr{C}|.$$

However, $\mathscr{C}$ lacks the terminal object, as well as the subobject classifier: the only constant presheaf contained in $\mathscr{C}$ is the empty one – the initial object. $\mathscr{C}$ inherits from $\mathscr{A}$ the logical structure of subobjects, but *not the higher order*.

The lattices of subobjects in $\mathscr{A}$, and hence in $\mathscr{C}$, are complete atomic Boolean algebras. In other words $\mathscr{A}$ is an *atomic* topos [7]. The atomic subobjects of a 3-coloured graph are just its *connected components*. An obvious way to extend the topological notion of component to colour algebras is to say that a colour subalgebra $B \subseteq A$ is a component of A if for all $x, y \in B$ there is a derived operation $\mathbf{f}$ such that $\mathbf{f}(x) = y$. By this very definition, the only proper subalgebra of a component is the empty one. Hence, components are the atoms.

6. 3-colourings as coalgebras

Routine calculations show that the functor $U: \mathscr{A} \to \mathscr{G}$ preserves all kinds of limits and colimits – with the exception of products. Indeed, a product of two cubic graps is by no means cubic. So U cannot have a left adjoint. But since it preserves colimits, it does have a right adjoint (for quite general reasons: e.g., [6, IV.1.6]). So we have

$$U \dashv R: \mathscr{G} \longrightarrow \mathscr{A}. \tag{5}$$

[3] This term has also been used by Bénabou in [9].

In fact, R can be rather effectively calculated, since $\mathscr{A}$ has that simple generating object Υ (described at the beginning of Section 4). Seen as a graph, Υ is the infinite *binary tree* with an infinite root, and with 3-coloured edges. Its vertices are finite strings of **r**, **b** and **g**, in which no immediate repetitions of a symbol are allowed.

Given a graph G, a 3-colouring RG will be obtained as a quotient of the coproduct

$$R'G := \coprod_{\mathscr{G}(U\Upsilon, G)} \Upsilon. \tag{6}$$

Since U preserves colimits, $UR'G$ is the coproduct of $\mathscr{G}(U\Upsilon, G)$ copies of $U\Upsilon$. By the couniversal property of the coproduct, there is a unqiue arrow

$$\varepsilon'_G : UR'G \longrightarrow G \tag{7}$$

such that every $f: U\Upsilon \to G$ factorizes as $f = \varepsilon'_G \circ \kappa_f$, where $\kappa_f : U\Upsilon \to UR'G$ is the fth injection into this coproduct. The object RG itself is now obtained by coequalizing all the endomorphisms of $R'G$ which commute with ε'_G. More precisely, if

$$Q = \{g : R'G \to R'G \mid \varepsilon'_G \circ Ug = \varepsilon'_G\}, \tag{8}$$

then RG comes about in the following coequalizer.

$$\coprod_Q R'G \underset{[g]_{g \in Q}}{\overset{[id]_{g \in Q}}{\rightrightarrows}} R'G \longrightarrow RG. \tag{9}$$

Since all parts of the above construction are functorial, it is now straightforward to spell out the arrow part of R. The counit $\varepsilon_G : URG \to G$ of the adjunction $U \dashv R$ is the factorisation of the arrow ε' through coequalizer (9).

To see that R is right adjoint to U, notice that RG is also the colimit of the larger diagram

$$\Pi : U/G \to \mathscr{A}, \tag{10}$$

which is the first projection from the comma category U/G. To see this, first observe that Υ is a generator of $\mathscr{A}$. (This means that the covariant functor represented by Υ is faithful.) Therefore, the subdiagram $\Lambda \subseteq \Pi$, spanned by those objects f of U/G which are in the form $f: U\Upsilon \to G$ must be cofinal with the whole Π. But the definition of RG, given above, is just a calculation of the colimit of Λ – and hence of Π.

We leave it to the reader to spell out the unit $\eta_A : A \to RUA$ of the adjunction $U \dashv R$. For every A, the arrow η_A is certainly monic, since the functor U is faithful. On the other side, the counit ε_G must be epi for all $G \in |\mathscr{G}|$, because $U\Upsilon$ is a generating object in $\mathscr{G}$ (which just means that ε'_G is epi). Therefore, the right adjoint R must be faithful too.

At any rate, the functor $H := UR$ is a cotriple on $\mathscr{G}$. So there is a comparison functor

$$\Phi : \mathscr{A} \longrightarrow \mathscr{G}^H, \tag{11}$$

where $\mathscr{G}^H$ is the category of H-coalgebras. Φ represents each colour algebra A as H-coalegbra $U\eta_A : UA \to HUA$. It has a right adjoint

$$\Psi : \mathscr{G}^H \longrightarrow \mathscr{A}, \tag{12}$$

which takes a coalgebra $\gamma : G \to HG$ to the equalizer of $R\gamma$ and η_{RG}.

Since U is faithful, it reflects monos and epis; but $\mathscr{A}$ is balanced, hence U reflects isos. Moreover, $\mathscr{A}$ has equalizers and U preserves them. Putting this together with the existence of a right adjoint to U, we conclude (by the dual of the Crude Tripleability Theorem [8, 3.5.]) that Φ and Ψ present an equivalence of categories. With no loss, $\mathscr{A}$ can be replaced by the category of coalgebras $\mathscr{G}^H$.

What happens with 3-colourings in this passage? When restricted to the subcategory $\mathscr{C} \hookrightarrow \mathscr{A}$ of 3-colourings, the functor U lands on bridgeless, loopless graphs in $\mathscr{G}$. Loopless graphs form a partial topos in $\mathscr{G}$ in a very much the same way as 3-colourings in $\mathscr{A}$. Bridgeless graphs, on the other hand, do not enjoy such strong closure properties. Nevertheless, even when restricted just to the partial topos $\mathscr{L} \hookrightarrow \mathscr{G}$ of loopless graphs – which does include graphs with bridges – the functor R still lands in the partial topos $\mathscr{C}$. (We shall see in Section 8 what happens with the bridges.) Putting this together, we get the adjunction

$$\tilde{U} \dashv \tilde{R} : \mathscr{L} \longrightarrow \mathscr{C} \tag{13}$$

as a restriction of $U \dashv R$. The cotriple $H : \mathscr{G} \to \mathscr{G}$ restricts to the cotriple $\tilde{H} : \mathscr{L} \to \mathscr{L}$. The category $\mathscr{C}$ of 3-coloured graphs is equivalent to the category of $\tilde{H}$-coalgebras on loopless graphs: the equivalence is realized by the appropriate restrictions of Φ and Ψ. Every 3-colouring can thus be presented as an $\tilde{H}$-coalgebra on a bridgeless, loopless graph.

In this context, the 4-Colour Theorem becomes the statement that

every cubic planar graph with no loops or bridges allows an H-coalgebra structure.

This structure is the edge 3-colouring.

7. On cofree 3-colourings

Now one wonders, of course, how do the constructed functors actually colour particular graphs. Moreover, if a graph G is not 3-colourable, what does the corresponding 3-coloured graph RG look like?

Let us take another look at the definition of the functor R. It has already been explained (in Section 6) that RG is the colimit of the diagram Λ in $\mathscr{A}$, which assigns a copy of $\hat{\Upsilon}$ to each $f \in \mathscr{G}(U\hat{\Upsilon}, G)$. An endomorphism $\alpha \in \mathscr{A}(\hat{\Upsilon}, \hat{\Upsilon})$ belongs to the diagram Λ as the arrow $\Lambda\alpha : \Lambda f \to \Lambda g$ if and only if $f = g \circ U\alpha$. On the other hand, the Yoneda embedding

$$\widehat{(-)} : \Upsilon \to \mathscr{A}$$

identifies the endomorphisms of the presheaf $\hat{\Upsilon}$ with the elements of the group Υ: since $\widehat{(-)}$ is full and faithful, each $\alpha \in \mathscr{A}(\hat{\Upsilon}, \hat{\Upsilon})$ appears as the image $\hat{\mathbf{a}}$ of a unique $\mathbf{a} \in \Upsilon$. Therefore, each α is an iso; all the arrows in the diagram Λ are isos.

But a *connected* diagram consisting of isomorphisms can be reduced to any of its full subdiagrams on a single object: the (co)limit will not change. On the other hand, the (co)limit of *any* diagram can be calculated as the (co)product of the (co)limits of its connected components. So we get the following picture of the colour algebra $RG = \operatorname{colim} \Lambda$.

Each of its connected components is a quotient of the 3-coloured binary tree Υ. Such a component corresponds to a connected component of the diagram Λ, i.e., to an equivalence class with respect to the binary relation $\sim$ on $\mathscr{G}(U\hat{\Upsilon}, G)$, defined:

$$f \sim g \iff \exists \mathbf{a} \in \Upsilon.\, f = g \circ U\hat{\mathbf{a}}. \tag{14}$$

The component of RG corresponding to the equivalence class of f can be obtained from Υ by identifying vertices $\mathbf{x}$ and $\mathbf{y}$ in it along the relation $\approx$, defined:

$$\mathbf{x} \approx \mathbf{y} \iff \exists \mathbf{a} \in \Upsilon.\, f = f \circ U\hat{\mathbf{a}} \wedge \mathbf{x} = \mathbf{a}\mathbf{y}. \tag{15}$$

Relations (14) and (15) together are not much more than a presentation of quotient (9). At a closer look, however, they provide a rather concrete picture of RG.

Each subgraph B of G, which can be obtained as a quotient of the tree $U\Upsilon$, contributes a number of components of RG. (A single vertex, or a disconnected graph cannot be obtained as a quotient of $U\Upsilon$.) These components correspond to the $\sim$-classes of graph epimorphisms $U\Upsilon \to B$. Each of them is projected by ε_G onto the whole subgraph B. If B can be 3-coloured, the underlying graphs of some of these components will be *isomorphic* to B. Moreover, *every* 3-colouring of B – each $C \in \mathscr{C}$ such that $UC = B$ – will appear as a component of RG. It will correspond to the $\sim$-class of Up, where $p \in \mathscr{C}(\Upsilon, C)$ is, of course, a colour preserving graph morphism. Given a 3-colouring C of B, such a representing morphism p is easily obtained. Since Υ is the free colour algebra over one generator and all the operations in colour algebras are bijective, one only needs to choose in C the image $p(\mathbf{x})$ of an arbitrary vertex $\mathbf{x}$ of Υ: this choice will completely determine the morphism p. Moreover, every two morphisms p obtained in this way will obviously be $\sim$-related.

And so, if the whole G can be 3-coloured, all its different colourings will be contained among the components of RG. An H-coalgebra $\gamma : G \hookrightarrow HG = URG$ will identify G with one of its coloured copies in RG.

8. Examples

In the end, let us take a look at some 3-colourings RG induced by graphs G which cannot be 3-coloured.

1. Let I be the graph (Fig. 3) with two vertices and one edge between them. Recalling that the vertices of Υ are words made of $\mathbf{r}$; $\mathbf{b}$ and $\mathbf{g}$, note that a graph

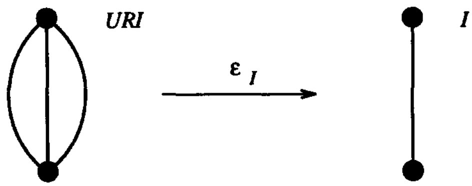

Fig. 3.

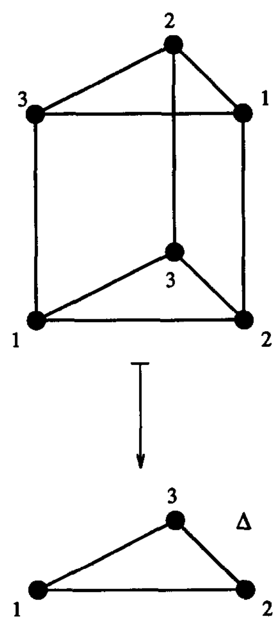

Fig. 4.

morphism $U\Upsilon \to I$ must send all the even-length words to one vertex of I, all the odd-length words to the other one. There are just two such morphisms, and they are $\sim$-equivalent. Therefore, RI will have just one component. This is the smallest 3-colourable graph, with just 2 vertices and 3-edges.

2. The smallest 3-colourable graph that covers a triangle Δ is the 3-sided prism (Fig. 4). Besides this prism, $R\Delta$ also contains copies of RI, projected on one-edge subgraphs of Δ, and the 3-colourings of cube, which ε_Δ projects on the two-edge subgraphs of Δ (Fig. 5).

3. The graph Ξ, depicted in Fig. 6, cannot be 3-coloured because it contains a brige (between vertices 3 and 4). Two components of the graph $UR\Xi$ which cover all of Ξ are shown on the left.

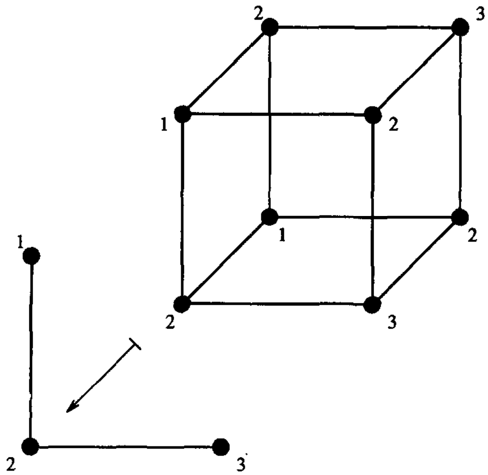

Fig. 5.

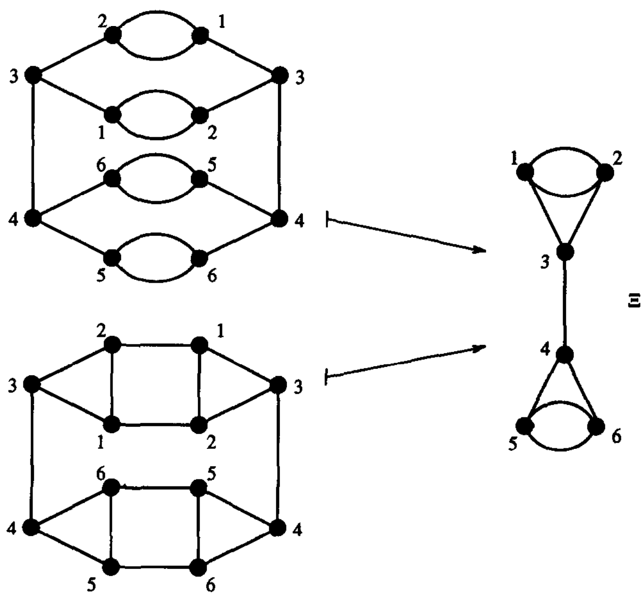

Fig. 6.

4. Finally, let us take another look at Petersen's graph P, from Fig. 1. A calculation of the relevant part of RP shows that the smallest 3-colouring which covers P is given on icosahedron (Fig. 7).

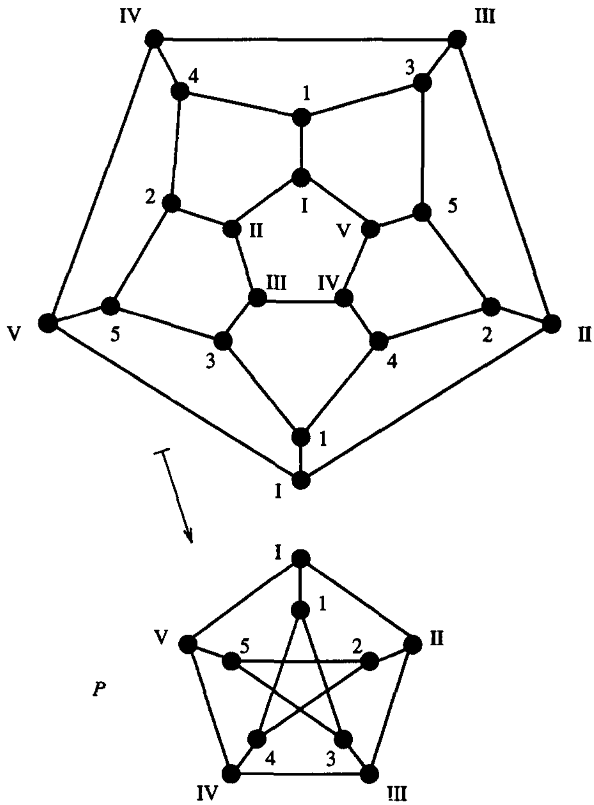

Fig. 7.

9. Conclusions?

In principle, the combinatorial approach to the 4-colouring hinges upon the induction principle – in one way or another. (Cf. [15].) Roughly, one assumes that all maps smaller than M can be 4-coloured, and then derives that this must be the case with M too. The proof provided by Appel and Haken is actually set up as a contraposition of the inductive step: the "minimal counterexample" is the smallest graph for which the inductive step cannot be made.

On the other hand, the categorical approach outlined here suggests that the 3-coloured graphs can be approached not only "from below", by induction over the smaller graphs, but also "from above", by projecting the coloured tree Υ and its quotients contained in cofree 3-colourings. Every 3-colouring of a given graph appears as a colimit of coalgebras.

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References

[1] M. Aigner, Graphentheorie. Eine Entwicklung aus dem 4-Farben Problem, Teubner-Studienbücher: Mathematik (Teubner, 1984).
[2] K. Appel and W. Haken, Every planar map is four colorable. Part I: Discharging, Illinois J. Math. 21 (1977) 429–490.
[3] K. Appel, W. Haken and J. Koch, Every planar map is four colorable. Part II: Reducibility, Illinois J. Math. 21 (1977) 491–567.
[4] K. Appel and W. Haken, The four color proof suffices, Math. Intelligencer 8/1 (1986) 20.
[5] K. Appel and W. Haken, Every Planar Map is Four Colorable, Contemporary Mathematics, Vol. 98 (Amer. Math. Soc., Providence, RI, 1989).
[6] M. Artin, A. Grothendieck and J.L. Verdier, Théorie des Topos et Cohomologie Etale des Schemas (*SGA*4), Lecture Notes in Mathematics, Vol. 269 (Springer, Berlin, 1972).
[7] M. Barr and R. Diaconescu, Atomic toposes, J. Pure Appl. Algebra 17 (1980) 1–24.
[8] M. Barr and C. Wells, Toposes, Triples and Teories, Grundlehren der math. Wis. 278 (Springer, Berlin, 1985).
[9] J. Bénabou, Fibered categories, 1983, manuscipt.
[10] B. Fawcett, A categorical characterisation of the Four Colour Theorem, Canad. Math. Bull. 29(4) (1986) 426–431.
[11] P.J. Freyd and A. Scedrov, Categories, Allegories, North-Holland Mathematical Library 39 (North-Holland, Amsterdam, 1990).
[12] P.T. Johnstone, Topos Theory, L.M.S. Mathematical Monographs, Vol. 10 (Academic Press, New York, 1977).
[13] A. Joyal, Une théorie combinatoire des séries formelles, Adv. Math. 42 (1981) 1–83.
[14] D. König, Theorie der endlichen und unendlichen Graphen (Akademische Verlagsgesellschaft, Leipzig, 1936; Chelsea, New York, 1950).
[15] D. Pavlović, A survey of recoloring, in preparation.
[16] T.L. Saaty and P.C. Kainen, The Four-Color Problem (McGraw-Hill, New York, 1977).
[17] W.T. Tutte, Graph Theory, Encyclopedia of Mathematics and its Applications 21 (Addison–Wesley, Reading, MA, 1984).
[18] H. Whitney and W.T. Tutte, Kempe chains and the four color problem, Utilitas Math. 2 (1972) 241–281.